17.1 Complex Numbers

Introduction You have undoubtedly encountered complex numbers in your earlier courses in mathematics. When you first learned to solve a quadratic equation $ax^2 + bx + c = 0$ by the quadratic formula, you saw that the roots of the equation are not real; that is, complex, whenever the discriminant $b^2 - 4ac$ is negative. So, for example, simple equations such as $x^2 + 5 = 0$ and $x^2 + x + 1 = 0$ have no real solutions. For example, the roots of the last equation are $-\frac{1}{2} + \frac{\sqrt{-3}}{2}$ and $-\frac{1}{2} + \frac{\sqrt{-3}}{2}$. If it is assumed that $\sqrt{-3} = \sqrt{3}\sqrt{-1}$, then the roots are written $-\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1}$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1}$.

A Definition Two hundred years ago, around the time that complex numbers were gaining some respectability in the mathematical community, the symbol i was originally used as a disguise for the embarrassing symbol $\sqrt{-1}$. We now simply say that i is the **imaginary unit** and define it by the property $i^2 = -1$. Using the imaginary unit, we build a general complex number out of two real numbers.

Definition 17.1.1 Complex Number

A **complex number** is any number of the form $z = a + ib$ where a and b are real numbers and i is the imaginary unit.

Terminology The number i in Definition 17.1.1 is called the **imaginary unit**. The real number x in $z = x + iy$ is called the **real part** of z; the real number y is called the **imaginary part** of z. The real and imaginary parts of a complex number z are abbreviated Re(z) and Im(z), respectively. For example, if $z = 4 - 9i$, then Re(z) = 4 and Im(z) = −9. A real constant multiple of the imaginary unit is called a **pure imaginary number**. For example, $z = 6i$ is a pure imaginary number. Two complex numbers are **equal** if their real and imaginary parts are equal. Since this simple concept is sometimes useful, we formalize the last statement in the next definition.

Note: The imaginary part of $z = 4 - 9i$ is -9 **not** $-9i$.

Definition 17.1.2 Equality

Complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are **equal**, $z_1 = z_2$, if $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.

A complex number $x + iy = 0$ if $x = 0$ and $y = 0$.

Arithmetic Operations Complex numbers can be added, subtracted, multiplied, and divided. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, these operations are defined as follows.

Addition: $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

Subtraction: $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

Multiplication:
$$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= x_1x_2 - y_1y_2 + i(y_1x_2 + x_1y_2)$$

Division:
$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$
$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}$$

The familiar commutative, associative, and distributive laws hold for complex numbers.

Commutative laws: $$\begin{cases} z_1 + z_2 = z_2 + z_1 \\ z_1 z_2 = z_2 z_1 \end{cases}$$

Associative laws: $$\begin{cases} z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \\ z_1(z_2 z_3) = (z_1 z_2) z_3 \end{cases}$$

Distributive law: $$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

In view of these laws, there is no need to memorize the definitions of addition, subtraction, and multiplication. To add (subtract) two complex numbers, we simply add (subtract) the corresponding real and imaginary parts. To multiply two complex numbers, we use the distributive law and the fact that $i^2 = -1$.

EXAMPLE 1 Addition and Multiplication

If $z_1 = 2 + 4i$ and $z_2 = -3 + 8i$, find **(a)** $z_1 + z_2$ and **(b)** $z_1 z_2$.

Solution **(a)** By adding the real and imaginary parts of the two numbers, we get

$$(2 + 4i) + (-3 + 8i) = (2 - 3) + (4 + 8)i = -1 + 12i.$$

(b) Using the distributive law, we have

$$\begin{aligned} (2 + 4i)(-3 + 8i) &= (2 + 4i)(-3) + (2 + 4i)(8i) \\ &= -6 - 12i + 16i + 32i^2 \\ &= (-6 - 32) + (16 - 12)i = -38 + 4i. \end{aligned}$$ ≡

There is also no need to memorize the definition of division, but before discussing that we need to introduce another concept.

Conjugate If z is a complex number, then the number obtained by changing the sign of its imaginary part is called the **complex conjugate** or, simply, the **conjugate** of z. If $z = x + iy$, then its conjugate is

$$\bar{z} = x - iy.$$

For example, if $z = 6 + 3i$, then $\bar{z} = 6 - 3i$; if $z = -5 - i$, then $\bar{z} = -5 + i$. If z is a real number, say $z = 7$, then $\bar{z} = 7$. From the definition of addition it can be readily shown that the conjugate of a sum of two complex numbers is the sum of the conjugates:

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2.$$

Moreover, we have the additional three properties

$$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

The definitions of addition and multiplication show that the sum and product of a complex number z and its conjugate $\bar{z}$ are also real numbers:

$$z + \bar{z} = (x + iy) + (x - iy) = 2x \tag{1}$$

$$z\bar{z} = (x + iy)(x - iy) = x^2 - i^2 y^2 = x^2 + y^2. \tag{2}$$

The difference between a complex number z and its conjugate $\bar{z}$ is a pure imaginary number:

$$z - \bar{z} = (x + iy) - (x - iy) = 2iy. \tag{3}$$

Since $x = \text{Re}(z)$ and $y = \text{Im}(z)$, (1) and (3) yield two useful formulas:

$$\text{Re}(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad \text{Im}(z) = \frac{z - \bar{z}}{2i}.$$

However, (2) is the important relationship that enables us to approach division in a more practical manner: To divide z_1 by z_2, we multiply both numerator and denominator of z_1/z_2 by the conjugate of z_2. This will be illustrated in the next example.

EXAMPLE 2 Division

If $z_1 = 2 - 3i$ and $z_2 = 4 + 6i$, find **(a)** $\dfrac{z_1}{z_2}$ and **(b)** $\dfrac{1}{z_1}$.

Solution In both parts of this example we shall multiply both numerator and denominator by the conjugate of the denominator and then use (2).

(a)

$$\frac{2-3i}{4+6i} = \frac{2-3i}{4+6i}\,\frac{4-6i}{4-6i} = \frac{8-12i-12i+18i^2}{16+36}$$

$$= \frac{-10-24i}{52} = -\frac{5}{26} - \frac{6}{13}i.$$

(b)

$$\frac{1}{2-3i} = \frac{1}{2-3i}\,\frac{2+3i}{2+3i} = \frac{2+3i}{4+9} = \frac{2}{13} + \frac{3}{13}i.$$

≡

Geometric Interpretation A complex number $z = x + iy$ is uniquely determined by an *ordered pair* of real numbers (x, y). The first and second entries of the ordered pairs correspond, in turn, with the real and imaginary parts of the complex number. For example, the ordered pair $(2, -3)$ corresponds to the complex number $z = 2 - 3i$. Conversely, $z = 2 - 3i$ determines the ordered pair $(2, -3)$. In this manner we are able to associate a complex number $z = x + iy$ with a point (x, y) in a coordinate plane. But, as we saw in Section 7.1, an ordered pair of real numbers can be interpreted as the components of a vector. Thus, a complex number $z = x + iy$ can also be viewed as a *vector* whose initial point is the origin and whose terminal point is (x, y). The coordinate plane illustrated in **FIGURE 17.1.1** is called the **complex plane** or simply the ***z*-plane**. The horizontal or x-axis is called the **real axis** and the vertical or y-axis is called the **imaginary axis**. The length of a vector z, or the distance from the origin to the point (x, y), is clearly $\sqrt{x^2 + y^2}$. This real number is given a special name.

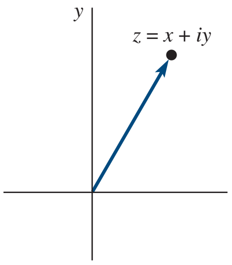

FIGURE 17.1.1 z as a position vector

Definition 17.1.3 Modulus or Absolute Value

The **modulus** or **absolute value** of $z = x + iy$, denoted by $|z|$, is the real number

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}. \tag{4}$$

EXAMPLE 3 Modulus of a Complex Number

If $z = 2 - 3i$, then $|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$. ≡

As **FIGURE 17.1.2** shows, the sum of the vectors z_1 and z_2 is the vector $z_1 + z_2$. For the triangle given in the figure, we know that the length of the side of the triangle corresponding to the vector $z_1 + z_2$ cannot be longer than the sum of the remaining two sides. In symbols this is

$$|z_1 + z_2| \le |z_1| + |z_2|. \tag{5}$$

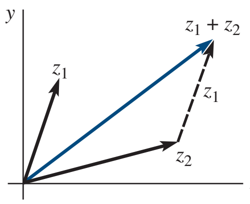

FIGURE 17.1.2 Sum of vectors

The result in (5) is known as the **triangle inequality** and extends to any finite sum:

$$|z_1 + z_2 + z_3 + \cdots + z_n| \le |z_1| + |z_2| + |z_3| + \cdots + |z_n|. \tag{6}$$

Using (5) on $z_1 + z_2 + (-z_2)$, we obtain another important inequality:

$$|z_1 + z_2| \ge |z_1| - |z_2|. \tag{7}$$

Remarks

Many of the properties of the real system hold in the complex number system, but there are some remarkable differences as well. For example, we cannot compare two complex numbers $z_1 = x_1 + iy_1$, $y_1 \neq 0$, and $z_2 = x_2 + iy_2$, $y_2 \neq 0$, by means of inequalities. In other words, statements such as $z_1 < z_2$ and $z_2 \geq z_1$ have no meaning except in the case when the two numbers z_1 and z_2 are real. We can, however, compare the absolute values of two complex numbers. Thus, if $z_1 = 3 + 4i$ and $z_2 = 5 - i$, then $|z_1| = 5$ and $|z_2| = \sqrt{26}$, and consequently $|z_1| < |z_2|$. This last inequality means that the point $(3, 4)$ is closer to the origin than is the point $(5, -1)$.